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CLOSED SEPARATOR SETS

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A *smallest separator* in a finite, simple, undirected graph G is a set $S \subseteq V(G)$ such that $G-S$ is disconnected and $|S|=\kappa(G)$, where $\kappa(G)$ denotes the *connectivity* of G.

A set $\mathcal{S}$ of smallest separators in G is defined to be *closed* if for every pair $S,T \in \mathcal{S}$, every component C of $G-S$, and every component D of $G-T$ intersecting C either $X(C,D) := (V(C)\cap T)\cup(T\cap S)\cup(S\cap V(D))$ is in $\mathcal{S}$ or $|X(C,D)| > \kappa(G)$. This leads, canonically, to a closure system on the (closed) set of all smallest separators of G.

A graph H with $V(H)\subseteq V(G)$, $E(H)\cap E(G)=\emptyset$ is defined to be $\mathcal{S}$-*augmenting* if no member of $\mathcal{S}$ is a smallest separator in $G\cup H := (V(G)\cup V(H), E(G)\cup E(H))$. It is proved that if $\mathcal{S}$ is closed then every minimally $\mathcal{S}$-augmenting graph is a forest, which generalizes a result of Jordán.

Several applications are included, among them a generalization of a Theorem of Mader on disjoint fragments in critically k-connected graphs, a Theorem of Su on highly critically k-connected graphs, and an affirmative answer to a conjecture of Su on disjoint fragments in contraction critically k-connected graphs of maximal minimum degree.

1. Introduction

All *graphs* considered here are supposed to be finite, undirected, and simple. For notation not defined here I would like to refer to [3] or [4]. The *connectivity* of a graph G is defined by $\kappa(G) := \min\{|T| : T \subseteq V(G)$ and $G-T$ is not connected$\}$ if G is noncomplete and $\kappa(G) := \max\{|V(G)|-1, 1\}$ if G is complete. For any natural number $k \le \kappa(G)$, G is called *k-connected*. A set $T \subseteq V(G)$ of cardinality $\kappa(G)$ such that $G-T$ is not connected is called

Mathematics Subject Classification (2000): 05C40 a *smallest separator*, and the set of all smallest separators of G is denoted by $\mathcal{T}(G)$.

The structure of the entire set $\mathcal{T}(G)$ has been studied in depth within the last three or four decades. In several parts of this theory, only those members of $\mathcal{T}(G)$ which have a certain fixed property as, for example, containing two adjacent vertices, are of interest. A typical application of this method is one of Thomassen's proofs of Kuratowski's Theorem [20].

A first way of generalizing results for $\mathcal{T}(G)$ systematically to certain subsets of $\mathcal{T}(G)$ has been worked out in [16]. This point of view describes the "interesting" separators by means of "containing certain substructures" of G, which then leads to a fruitful concept of generalized critical connectivity.

Here we take a different approach, by studying subsets of $\mathcal{T}(G)$ which have a certain additional property, namely to be *closed*, as it is defined in the abstract. It leads to natural concepts of generalized *higher* or *local* criticity, as they have been designed some years ago in order to solve several problems on highly critically connected graphs, but also to a generalization of the notion of *almost critical* graphs.

Beside the generalizations of older results on critically connected graphs or the more recent ones on connectivity augmentation, we present, as applications, some originary new results on size and distribution of fragments (see next section) in certain graphs. For example, generalizing the respective recent results on k-connected graphs for $k \in \{4,5,6\}$ to arbitrary k, we prove that the number of vertices of degree at most $5k/4-1$ in a graph without a k-contractible edge (i. e. an edge whose contraction produces a k-connected graph) grows lineary in the order of G. Noncomplete graphs without k-contractible edges are called *contraction critically k-connected*. We improve, moreover, a result of Su on the existence of two small disjoint fragments in these graphs, and prove his conjecture on the existence of six disjoint fragments in any contraction critically k-connected graph of minimum degree at least $\frac{5}{4}k-1$.

2. Closed separator sets

Let G be a graph and $\mathcal{S} \subseteq \mathcal{T}(G)$. For $T \in \mathcal{S}$, the union of the vertex sets of at least one but not of all components of $G-T$ is called a *$T-\mathcal{S}$-fragment* or, briefly, an $\mathcal{S}$-fragment. For $F \subseteq V(G)$, let $\overline{F}^G := V(G)-(F \cup N_G(F))$, where $N_G(F) := \{x \in V(G)-F : x \text{ is adjacent to some vertex in } F\}$ denotes the *neighborhood* of F in G. So F is an $\mathcal{S}$-fragment if and only if $F \neq \emptyset$, $\overline{F}^G \neq \emptyset$, and $N_G(F) \in \mathcal{S}$, and F is an $\mathcal{S}$-fragment if and only $\overline{F}^G$ is. An inclusion minimal $\mathcal{S}$-fragment is called an $\mathcal{S}$-*end*, and an $\mathcal{S}$-fragment of minimum cardinality is called an $\mathcal{S}$-*atom*. A $T-\mathcal{S}$-*end* is an $\mathcal{S}$-end F such that $N_G(F)=T$, a $T-\mathcal{S}$-*atom* is an $\mathcal{S}$-atom F such that $N_G(F)=T$. If $\mathcal{S}=\mathcal{T}(G)$ then we omit the symbol $\mathcal{S}$, thus defining the concepts of T-*fragments*, T-*ends*, T-*atoms*, *fragments*, *ends*, and *atoms*, respectively. If F is a fragment then $\overline{F}^G$ is called its *complementary fragment*. A fragment F is called *proper* if $|F|\leq|\overline{F}^G|$. We omit the index G in the notation $\overline{F}^G$ if it is clear from the context (which will be almost always the case).

An $S\in\mathcal{T}(G)$ *crosses* a $T\in\mathcal{T}(G)$ if S intersects every component of $G-T$ (or, equivalently, T intersects every component of $G-S$, or, equivalently, S intersects two components of $G-T$).

We start with a fundamental property of fragments in graphs. Although proofs can be found in many papers cited in the literature list, I added one for keeping this paper self-contained.

Lemma 1. *For two fragments F, F' in a graph G, let*

$$X_G(F,F') := (F\cap N_G(F'))\cup(N_G(F)\cap N_G(F'))\cup(N_G(F)\cap F').$$

If $F\cap F'\neq\emptyset$ then

$$|F\cap N_G(F')|\geq|\overline{F'}\cap N_G(F)|, \tag{1}$$

and if equality holds in (1) then $F\cap F'$ is a $X_G(F,F')$-fragment. In particular, $F\cap F'$ and $\overline{F}\cap\overline{F'}$ are both fragments if and only if they are both not empty.

Proof. Clearly, $N_G(F\cap F')\subseteq X_G(F,F')=(F\cap N_G(F'))\cup(N_G(F)-\overline{F'})$. It follows $k\leq|N_G(F\cap F')|\leq|X_G(F,F')|=|F\cap N_G(F')|+|N_G(F)-\overline{F'}|=|F\cap N_G(F')|+k-|\overline{F'}\cap N_G(F)|$, which implies (1). Furthermore, if equality holds then $N_G(F\cap F')=X_G(F,F')$. Applying these assertions to $\overline{F'},\overline{F}$ for F,F', the last statement follows as well. ∎

The estimation (1) will be used widely later, and instead of refering to Lemma 1 at the appropriate places I rather write $\geq^*$, $=^*$, $>^*$, $\subseteq^*$, etc.

A variety of arguments and results on fragments in graphs rely – explicitely or hidden – on the statements of Lemma 1. The following definition takes care of this observation. For some fixed graph G, let's call a set $\mathcal{S}\subseteq\mathcal{T}(G)$ a *closed* separator set in the graph G, if $F\cap F'$ is an $\mathcal{S}$-fragment for every pair F,F' of intersecting $\mathcal{S}$-fragments for which equality holds in (1).

If $\mathcal{S}\subseteq\mathcal{T}(G)$ consists of at most one separator then $\mathcal{S}$ is closed trivially, and $\mathcal{T}(G)$ itself is closed by Lemma 1. Since, by definition, intersections of closed separator sets are closed, this leads to a *closure* or *hull operator* for separator sets. For any $\mathcal{R} \subseteq \mathcal{T}(G)$ let $cl(\mathcal{R}) := \bigcap\{\mathcal{S} \supseteq \mathcal{R} \colon \mathcal{S} \subseteq \mathcal{T}(G) \text{ is closed}\}$ define the *closure* of $\mathcal{R}$.

In a graph G of connectivity $\kappa(G)=1$, every subset $\mathcal{S}$ of smallest separators is closed. Let's consider the supergraph H of such a graph G obtained by adding an edge between vertices x,y whenever there is no $\mathcal{S}$-fragment F such that $x \in F$ and $y \in \overline{F}$, i. e. whenever x,y are not separable by some $T \in \mathcal{S}$. It is not hard to see (and formally proved below) that the fragments of H are the $\mathcal{S}$-fragments of G. Informally, the fragments of H "simulate" the $\mathcal{S}$-fragments.

At this very early point of the considerations it thus seems to be worthwhile to have a look at the following question for general connectivity $\kappa \geq 1$. (An affirmative answer would have made several considerations below obsolete.) Given a (closed) set $\mathcal{S}$ of separators in a graph G of connectivity κ, is there a supergraph H of G with $V(H)=V(G)$ such that the fragments of H are the $\mathcal{S}$-fragments of G? (Since $\mathcal{T}(H)$ is closed, it's a sort of minimal condition to $\mathcal{S}$ to be closed as well.)

For $\kappa=1$ the answer is "yes", and so it is for $\kappa=2$, as it is proved below. For $\kappa \geq 3$, the answer is "no", as it is indicated by the following construction.

Consider two disjoint k-regular k-connected graphs G_1, G_2 such that for every fragment F of G_1 or G_2, either $|F|=1$ or $|\overline{F}|=1$ holds. Suppose, moreover, that G_j contains a set E_j of k independent edges such that $G - E_j$ is connected, $j \in \{1,2\}$. Such graphs exist for every $k \geq 3$ – take, for example, the hypercube Q_k for G_1 and G_2. (Here $V(Q_k)=\{0,1\}^k$ and $E(Q_k)=\{xy\colon x,y \in V(Q_k), h(x,y)=1\}$, where $h(x,y) := |\{j \in \{1,\ldots,k\} : x_j \neq y_j\}|$ denotes the *Hamming distance* of x and y.) Clearly, such graphs don't exist for $k \leq 2$.

Let G_j' be obtained from G_j by subdividing every edge in E_j once, and let $s_{j,1}, \ldots, s_{j,k}$ be the respective subdivision vertices. So $|V(G_j')| \geq 3k$. Let G' be obtained from G_1' and G_2' by identifying $s_{1,i}$ and $s_{2,i}$ for every $i \in \{1,\ldots,k\}$, and denote by S the set of the k vertices $s_{1,1}=s_{2,1},\ldots,s_{1,k}=s_{2,k}$. Let G be obtained from G' by adding all edges in between distinct vertices of S. It follows that $|V(G)| \geq 5k$. We will consider G_1', G_2' as subgraphs of G and $V(G_j)$ as a subset of $V(G_j')$ for $j \in \{1,2\}$. So $V(G)$ is the disjoint union of $V(G_1)$, $V(G_2)$, and S.

Before studying the connectivity properties of G, let me illustrate the construction by an example, where $k=3$ and G_1, G_2 are both isomorphic to $K_{3,3}$. The three vertices of degree 6 represent S (see Fig. 1).

Let's continue with the general example.

Claim 1. G is k-connected.

We will prove this by applying the well known theorem of Menger (see [3] or [4]). For $j \in \{1,2\}$ and $x \neq y \in V(G_j)$ there exist k openly disjoint x,y-paths

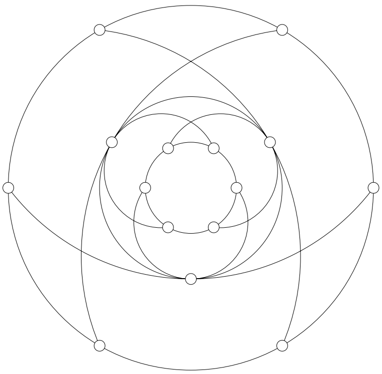

Fig. 1.

in G_j and, thus, in G'_j and G as well. Let $x_j \in V(G_j)$. Then we may choose for every edge $e \in E_j$ an endvertex $y_e \neq x_j$. Since G_j is k-connected, there exist k openly disjoint $x_j, \{y_e : e \in E_j\}$-paths in G_j and, thus, in G'_j. This easily transforms to a system of k openly disjoint x,S-paths in G'_j. Hence, for $x_1 \in V(G_1) - S$ and $x_2 \in V(G_2) - S$ there exist k openly disjoint x_1,x_2-paths. If there was a $T \subseteq V(G)$ such that $|T| < k$ and $G - T$ is not connected then all vertices of $V(G) - (S \cup T)$ would belong to the same component of $G - T$, and all other components would be contained in S. However, if S' is a nonempty subset of S then $|N_G(S')| \geq 4|S'| + k - |S'| \geq k+3$, so $|T| \geq k+3$, a contradiction. It follows that G cannot be separated by less than k vertices – so G is k-connected, proving Claim 1.

Let $\mathcal{S} := \{N_G(x) : x \in V(G) - S\}$.

Claim 2. $\mathcal{T}(G) = \mathcal{S} \cup \{S\}$.

Since every element of $\{S\} \cup \mathcal{S}$ separates G and has cardinality k, G has connectivity $\kappa(G) = k$ and $\{S\} \cup \mathcal{S} \subseteq \mathcal{T}(G)$.

Now consider a $T \neq S$ in $\mathcal{T}(G)$. Since $G(S)$ is complete, exactly one connected T-fragment F contains vertices of S, and T and S do not cross. Hence $V(G_j) \subseteq F$ for some $j \in \{1,2\}$, where we may assume $j = 1$ without loss of generality. Now take any vertex y in $\overline{F} \subseteq V(G_2)$. Since $S' := S \cap F \neq \emptyset$ has at least $2|S'| + |T - S'| = k + |S'| > k$ many neighbors in $V(G_2) \cup (S \cap T)$ and T does not intersect $V(G_1)$, not all neighbors of S' in $V(G_2)$ can be contained in T. Hence there exists a vertex $x \in F \cap V(G_2)$ adjacent to some $s' \in S'$. Note that T separates $x \in V(G'_2) - S$ from $y \in V(G'_2) - S$ in G'_2. Now let R be obtained from T by replacing every vertex in S by the endvertex of the corresponding edge in E_2 it subdivides which is contained in F. Then R separates x from y in G_2 and $|R| = |T| = k$. So R is the neighborhood of x or the neighborhood of y in G_2.

If R is the neighborhood of y in G_2 then T is the neighborhood of y in G, so $T \in \mathcal{S}$. If R is the neighborhood of x in G_2 then $|S'|=1$ and all $k-1$ vertices of $T \cap S$ have been replaced by the $k-1$ neighbors of x in $V(G_2)$. This implies, however, that $G_2 - E_2$ is not connected, a contradiction.

Hence $\mathcal{T}(G)=\{S\} \cup \mathcal{S}$, which proves Claim 2.

Claim 3. $\mathcal{S}$ is closed.

For consider two intersecting $\mathcal{S}$-fragments F, F'. If $|F|=1$ or $|F'|=1$ or $T=T'$ then $F \cap F'$ is one of F, F' and, thus, an $\mathcal{S}$-fragment. Otherwise, $F=\overline{\{x\}}$ and $F'=\overline{\{x'\}}$ for distinct x, x' in $V(G)-S$. Suppose that equality holds in (1). Then $|F \cap F'|=1$. Thus, $|V(G)| \leq |\overline{F}|+|\overline{F'}|+|N_G(F)|+|N(G(F')|+|F \cap F'| \leq 2k+3$, which contradicts $|V(G)| \geq 5k$. This proves Claim 3.

Consider an arbitrary proper supergraph H of G such that $V(H)=V(G)$. Since at least one vertex in $V(H)-S$ has degree at least $k+1$, it follows $\mathcal{T}(H) \neq \mathcal{S}$.

So, by the previous claims, $\mathcal{S}$ is a closed set of separators of the graph G, and G has no supergraph H with $V(H)=V(G)$ such that the fragments of H are the $\mathcal{S}$-fragments of G. This proves the proposed "no".

Taking these considerations into account, the connectivity bound in the following theorem cannot be increased.

Theorem 1. *Let $\mathcal{S}$ be a closed set of smallest separating sets of a graph G of connectivity at most* 2.

Let H be defined by $V(H) := V(G)$ and $E(H) := \{xy :$ there exists no $\mathcal{S}$-fragment F such that $x \in F$, $y \in \overline{F}\}$.

Then H is a supergraph of G whose fragments are the $\mathcal{S}$-fragments of G.

Proof. For brevity, let $\mathcal{T} := \mathcal{T}(H)$. Clearly, H is a supergraph of G, so $\kappa(H) \geq \kappa(G)$, and if $\mathcal{S} = \emptyset$ then H is complete and, thus, $\mathcal{T} = \emptyset$. Hence we may assume that $\mathcal{S} \neq \emptyset$.

Consider a $T-\mathcal{S}$-fragment F in G. Then $N_H(F) \supseteq N_G(F)=T$, and since no vertex in F is adjacent in H to a vertex in $\overline{F}^G$, $N_H(F)=T$. It follows that F is a fragment of H and that $\kappa(H) = \kappa(G)$. Consequently, $\mathcal{S} \subseteq \mathcal{T}$. Since every $\mathcal{S}$-fragment of H must is an $\mathcal{S}$-fragment of G as well, it suffices to prove $\mathcal{S}=\mathcal{T}$.

Suppose, to the contrary, that $\mathcal{R} := \mathcal{T} - \mathcal{S} \neq \emptyset$, and consider a $T_B - \mathcal{R}$-end B in H. For any fixed $x \in B$, let $\mathcal{F}'_x$ denote the set of all $\mathcal{S}$-fragments F' in H such that $x \in F'$ and $\overline{B} \cap \overline{F'} \neq \emptyset$. Since x can be separated from any $y \in \overline{B}$ by some member of $\mathcal{S}$, $\bigcup_{F' \in \mathcal{F}'_x} \overline{F'} \supseteq \overline{B}$ follows. In particular, $\mathcal{F}'_x \neq \emptyset$.

Claim 1. For every $x \in B$, there exists some $F' \in \mathcal{F}'_x$ with $B \not\subseteq F'$.

Let's assume, to the contrary, $B \subseteq F'$ for every $F' \in \mathcal{F}'_x$. There exists an inclusion minimal $F' \in \mathcal{F}'_x$. B is properly contained in F', and we may consider an $y \in (T' := N_H(F')) \cap \overline{B}$. Since y can be separated from x by some member of $\mathcal{S}$, there exists a $C \in \mathcal{F}'_x$ such that $y \in \overline{C}$. By assumption, $B \subseteq C$, so $D := F' \cap C$ contains B. If $|C \cap T'| = |\overline{F'} \cap N_H(C)|$ or $|F' \cap N_H(C)| = |\overline{C} \cap T'|$ then D would be an $\mathcal{S}$-fragment (by Lemma 1) properly contained in F' (since $\overline{D}$ properly contains $\overline{F'}$), thus violating the minimality of F'. Hence $|C \cap T'| >^* |\overline{F'} \cap N_H(C)|$ and $|F' \cap N_H(C)| >^* |\overline{C} \cap T'|$. It follows that T' intersects both C and $\overline{C}$, so $N_H(C)$ intersects both F' and $\overline{F'}$, which implies $2 \geq |N_H(C)| \geq |F' \cap N_H(C)| + |\overline{F'} \cap N_H(C)| > |\overline{C} \cap T'| + 1 \geq 2$, a contradiction. This proves Claim 1.

Claim 2. B is an end in H.

For otherwise there exist fragments in H properly contained in B, and we may choose F maximal among them. Then there exists an $x \in (T := N_H(F)) \cap B$. By Claim 1, there exists an $F' \in \mathcal{F}'_x$ with $B \not\subseteq F'$, hence $C := B \cap F'$ is a fragment properly contained in B. Since B is an $\mathcal{R}$-end, C is an $\mathcal{S}$-fragment properly contained in B. Hence $D := \overline{F} \cap \overline{C}$ contains $\overline{B}$. If $|\overline{C} \cap T| = |F \cap N_H(C)|$ then $\overline{D} = F \cup C$ would be an $\mathcal{S}$-fragment contained in B and properly containing F, violating the maximality of F. Hence $|\overline{C} \cap T| >^* |F \cap N_H(C)|$. Since $x \in C \cap T$, $N_H(C)$ separates T, and, thus, T separates $N_H(C)$. This implies $2 \geq |T| \geq |\{x\}| + |\overline{C} \cap T| > 1 + |F \cap N_H(C)| \geq 2$, which is absurd. This proves Claim 2.

Now take any $x \in B$. Again, by Claim 1, there exists an $F' \in \mathcal{F}'_x$ with $B \not\subseteq F'$, hence $B \cap F'$ is a fragment properly contained in B, contradicting Claim 2. ∎

To close this section, let's have a look at two elementary examples of closed separator sets.

Lemma 2. *Let $\mathcal{S}$ be a closed separator set in a graph G and let $X, Y \subseteq V(G)$. Then*

$$\mathcal{R} := \{T \in \mathcal{S} : X \subseteq T \subseteq V(G) - Y\}$$

is closed.

Proof. If F is a $T - \mathcal{R}$-fragment and F' is a $T' - \mathcal{R}$-fragment intersecting F such that equality holds in (1) then $X(F, F') \in \mathcal{S}$ since $\mathcal{S}$ is closed, and $X \subseteq T \cap T' \subseteq X(F, F') \subseteq T \cup T' \subseteq V(G) - Y$, which then implies $X(F, F') \in \mathcal{R}$. ∎

Lemma 3. *Let $\mathcal{S}$ be a closed separator set in a graph G and let B be an $\mathcal{S}$-end. Then*

$$\mathcal{R} := \{T \in \mathcal{S} : T \cap B \neq \emptyset\}$$

is closed.

Proof. Let F be a T–$\mathcal{R}$-fragment and let F' be a T'–$\mathcal{R}$-fragment intersecting F such that equality holds in (1). Then $X(F,F')\in\mathcal{S}$, and in order to show that $X(F,F')\in\mathcal{R}$, it suffices to prove $X(F,F')\cap B\neq\emptyset$.

The assertion is clear if $B\subseteq F\cup T$. Hence $\overline{F}\cap B\neq\emptyset$, so $F\cap\overline{B}=^*\emptyset$. The assertion is also clear if $F\cap F'\subseteq N_G(B)$. Hence $F\cap B\neq\emptyset$, implying $\overline{B}\subseteq^* T$. Symmetrically, $\overline{B}\subseteq T'$, so $\overline{B}\subseteq T\cap T'\subseteq X(F,F')$, which proves $X(F,F')\cap B\neq\emptyset$. ∎

3. Generalized augmentation

Given a graph G of connectivity k, a graph H with $V(H)\subseteq V(G)$ and $E(H)\cap E(G)=\emptyset$ is called *augmenting* for G if $G+H$ is $(k+1)$-connected. Of particular interest is the structure of "minimally augmenting graphs". It is known that these must be forests [6], and in this section, we shall generalize this result.

Expressed in terms of smallest separators, a graph H with $V(H)\subseteq V(G)$ and $E(H)\cap E(G)=\emptyset$ is augmenting for the noncomplete graph G if and only if $\mathcal{T}(G+H)\cap\mathcal{T}(G)=\emptyset$. To state this informally, one could say that every separator T of G is "bridged" or "eliminated" by some edges of H in the sense that for every T-fragment F of G there exists an edge $xy\in E(H)$ with $x\in F$ and $y\in\overline{F}$.

This reformulation now leads to a generalized concept of augmenting graphs. Let $\mathcal{S}$ be a set of smallest separators in a graph G. A graph H is called $\mathcal{S}$-*augmenting* for G if $V(H)\subseteq V(G)$, $E(H)\cap E(G)=\emptyset$, and $\mathcal{T}(G+H)\cap\mathcal{S}=\emptyset$. If, moreover, H has no proper subgraph H' with $V(H')=V(H)$ such that H' is $\mathcal{S}$-augmenting for G then H is called *minimally* $\mathcal{S}$-*augmenting*.

Clearly, every augmenting graph is also an $\mathcal{S}$-augmenting graph for G. Hence there always *exists* a minimally $\mathcal{S}$-augmenting graph for G which is a forest. However, in the central application of the next section it will be necessary to prescribe the vertex set of a potential minimally $\mathcal{S}$-augmenting subgraph H. Let us first consider an example showing that, without any further condition to $\mathcal{S}$, a minimally $\mathcal{S}$-augmenting graph need not to be a forest.

We realize the cycle C_8 of length 8 by $V(C_8)=\{0, 1, 2, 3, 4, 5, 6, 7\}$ and $E(C_8)=\{01, 12, 23, 34, 45, 56, 67, 70\}$, and we consider $\mathcal{S}:=\{\{0,3\}, \{2,5\}, \{4,7\}, \{6,1\}\}\subseteq\mathcal{T}(C_8)$. There are eight $\mathcal{S}$-fragments, namely $\{1,2\}$, $\{3,4\}$, $\{5,6\}$, $\{7,0\}$, and their complementary fragments. The reader is invited to check that the cycle $H=(\{1,3,5,7\},\{13,35,57,71\})$ is indeed a minimally $\mathcal{S}$-augmenting graph for C_8.

In our example, the $\mathcal{S}$-fragments $\{1,2\}$ and $\overline{\{3,4\}}$ satisfy (1) with equality, but their intersection, $\{1\}$, is not an $\mathcal{S}$-fragment. Hence $\mathcal{S}$ is not closed (try to determine its closure $cl(\mathcal{S})$).

We proceed with the main result of this section, stating that if $\mathcal{S}$ is closed then every minimally $\mathcal{S}$-augmenting graph is a forest.

We follow the lines of the proofs of Lemma 4 in [15] and the results of [13]. A remark of Mader in [15] indicates that these are closely related topics. In fact, they both meet in Theorem 2: Jordán proved its assertion for $\mathcal{S}=\mathcal{T}(G)$ by using results of [13], here we give a proof of the general statement using the following elementary but essential observation on fragments Lemma 4 in [15] depends upon.

Lemma 4. *If F,F' are fragments of some graph G such that $F\cap\overline{F'}\neq\emptyset$ and $\overline{F}\cap F'=\emptyset$ then $|F|>|F'|$.*

Proof. By Lemma 1, we immediately get $|F|>|F\cap F'|+|F\cap N_G(F')|\geq^* |F'\cap F|+|F'\cap N_G(F)| = |F'|$. ∎

Theorem 2. *Let $\mathcal{S}$ be a closed set of smallest separators in a graph G. Then every minimally $\mathcal{S}$-augmenting graph is a forest.*

Proof. Suppose, to the contrary, that there is a minimally $\mathcal{S}$-augmenting graph H for G which contains a cycle $x_0,x_1,x_2,\ldots,x_{\ell-1},x_0$.

For $j\in\{0,\ldots,\ell-1\}$, let e_j be the edge in H with $V(e_j)=\{x_j,x_{j+1}\}$ (indices mod ℓ). Since $H-e_j$ is not $\mathcal{S}$-augmenting, we may choose an $\mathcal{S}$-fragment F_j in $G+(H-e_j)$ such that $x_{j+1}\in F_j$ and $x_j\in\overline{F_j}$. Clearly, F_j is also an $\mathcal{S}$-fragment in G.

For "consecutive" fragments we obtain, first, $x_{j+1}\in F_j\cap\overline{F_{j+1}}$. Let's assume for a while that $X:=\overline{F_j}\cap F_{j+1}\neq\emptyset$, too. Then X is a fragment in G, and, since F_j,F_{j+1} are $\mathcal{S}$-fragments in $G+(H-\{e_j,e_{j+1}\})$, X is an $\mathcal{S}$-fragment in $G+(H-\{e_j,e_{j+1}\})$ as well. However, since $N_H(x_{j+1})\cap X=\emptyset$, X is also an $\mathcal{S}$-fragment in $G+H$, a contradiction.

Hence we found a system of fragments F_j such that $F_j\cap\overline{F_{j+1}}\neq\emptyset$ and $\overline{F_j}\cap F_{j+1}=\emptyset$ for $j\in\{0,\ldots,\ell-1\}$. By Lemma 4, $|F_0|>|F_1|>\ldots>|F_{\ell-1}|>|F_0|$, which is absurd. ∎

4. End covers and disjoint fragments

Let again $\mathcal{S}$ be a set of smallest separators in a graph G. A set $W\subseteq V(G)$ is called an *$\mathcal{S}$-end cover in G*, if it intersects every $\mathcal{S}$-end. Since W intersects every $\mathcal{S}$-end if and only if it intersects every $\mathcal{S}$-fragment, one could call an $\mathcal{S}$-end cover equally well an *$\mathcal{S}$-fragment cover* (cf. [7]).

The vertex set of any $\mathcal{S}$-augmenting graph for G is an $\mathcal{S}$-end cover, and, conversely, for every $\mathcal{S}$-end cover W, $K_W - E(G)$ is an $\mathcal{S}$-augmenting graph for G, where K_W denotes the complete graph on W. Hence the $\mathcal{S}$-end covers correspond to the vertex sets of $\mathcal{S}$-augmenting graphs for G.

According to [16], a noncomplete graph G of connectivity k is called *almost critically* k-connected if and only if every fragment of G is intersected by a smallest separating set. Hence a graph G of connectivity k is almost critically k-connected if and only if $\bigcup \mathcal{T}(G)$ is a $\mathcal{T}(G)$-end cover, and one is attempted to call a graph of connectivity k *almost critically k-connected with respect to* $\mathcal{S}$ if $\bigcup \mathcal{S}$ is an $\mathcal{S}$-end cover.

The following is the obvious generalization of the main result in [7] (which first has been proved in [18]).

Theorem 3. *Let ℓ be a natural number, $\mathcal{S}$ be a closed set of smallest separators in a graph G, let W be an $\mathcal{S}$-end cover, and suppose that for every set $U \subseteq W$ of cardinality at most ℓ there exists a $T \in \mathcal{S}$ containing U.*

Then $|W| \geq 2\ell+2$. In particular, G has at least $2\ell+2$ distinct $\mathcal{S}$-ends.

Proof. The proof is almost literally the same as the corresponding one in [7]. Suppose, to the contrary, that $|W| \leq 2\ell+1$. Let K_W be the complete graph on W and $K_W^- := K_W - E(G)$. Since W is an $\mathcal{S}$-end cover, K_W^- is an $\mathcal{S}$-augmenting graph for G. Thus, there exists a minimally $\mathcal{S}$-augmenting graph H for G such that $V(H) = W$. By Theorem 2, H is a forest, in particular, bipartite. Hence there exists a decomposition of $V(H)$ into two independent sets U, U', where $|U| \leq |U'|$. By assumption, $|U| \leq \ell$, and by the conditions to $\mathcal{S}$ in the assertion, $U \subseteq T$ holds for some $T \in \mathcal{S}$. However, T separates $G+H$, a contradiction. ∎

In Theorem 3, “distinct ends” cannot be replaced with “disjoint ends”, for there are almost critically k-connected graphs which do not admit a system of four disjoint fragments [16]. However, for $\ell = 2$ a stronger statement will be formulated, which generalizes Theorem 6 in [16] and Theorem 3 in [19]. We start with the following Lemma (cf. Lemma 3 in [15]).

Lemma 5. *Let $\mathcal{S}$ be a closed set of smallest separators in a graph G of connectivity k.*

Suppose that B is an $\mathcal{S}$-end and $T \in \mathcal{S}$ intersects B. Let F be a T-fragment.

Then one of the following is true.

1. $F \subseteq N_G(B)$ *and* $|\overline{F} \cap N_G(B)| > |F|$, *or*
2. $\overline{F} \subseteq N_G(B)$ *and* $|F \cap N_G(B)| > |\overline{F}|$, *or*
3. $B \subseteq T$ *and* $|\overline{B} \cap T| \geq |B|$, *or*

4. $\overline{B} \subseteq T$ *and* $|B \cap T| > |\overline{B}|$.

Proof. Let $T_B := N_G(B)$. Suppose first that $F \cap B \neq \emptyset$ holds. Then $\overline{F} \cap \overline{B} =^* \emptyset$.

If $\overline{F} \cap B \neq \emptyset$ then $F \cap \overline{B} =^* \emptyset$ and $|B| > |B \cap T| >^* |F \cap T_B| >^* |\overline{B} \cap T| = |\overline{B}|$, so 4. holds.

Hence we may assume that $\overline{F} \cap B = \emptyset$, so $\overline{F} \subseteq T_B$. If $F \cap \overline{B} \neq \emptyset$ then 2. holds, since $|F \cap T_B| >^* |\overline{B} \cap T| \geq^* |\overline{F} \cap T_B| = |\overline{F}|$. If, otherwise, $F \cap \overline{B} = \emptyset$ then $\overline{B} \subseteq T$. Then $|F \cap T_B| + |B \cap T| \geq^* |\overline{B} \cap T| + 1 + |\overline{F} \cap T_B| + 1 = |\overline{B}| + |\overline{F}| + 2$, so either 2. or 4. holds.

By symmetry, $\overline{F} \cap B \neq \emptyset$ implies that 1. or 4. is true. Hence we may assume that $B \subseteq T$. If $\overline{B} \subseteq T$, too, then one of 3. or 4. holds. Hence, without loss of generality, we may assume $F \cap \overline{B} \neq \emptyset$. If $\overline{F} \cap \overline{B} \neq \emptyset$ then $|\overline{B} \cap T| \geq^* |F \cap T_B| \geq^* |B \cap T| = |B|$, so 3. holds. If, otherwise, $\overline{F} \cap \overline{B} = \emptyset$ then $\overline{F} \subseteq T_B$, and $|F \cap T_B| + |\overline{B} \cap T| \geq |B \cap T| + |\overline{F} \cap T_B| = |B| + |\overline{F}|$. Hence either 2. or 3. holds. ∎

Suppose that an end B of a graph G without fragments of cardinality less than $k/2$ is intersected by some $T \in \mathcal{T}(G)$. Then Lemma 5 implies that $B \subseteq T$ and $|B| = k/2$ (and $|T \cap \overline{B}| = k/2$) holds. In particular, B is an atom. We keep this observation for a forthcoming reference.

Lemma 6. *Let G be an almost critically k-connected graph without fragments of cardinality less than $k/2$. Then every end of G has cardinality $k/2$.*

Let's have a look at another result obtained from Lemma 5.

Theorem 4. *Let $\mathcal{S}$ be a closed set of smallest separators in a graph G. Suppose that A is an $\mathcal{S}$-atom and $T \in \mathcal{S}$ intersects A.*

Then $A \subseteq T$ and $|A| \leq |N_G(A) - T|/2$.

Proof. Suppose first, to the contrary, that $F \cap A \neq \emptyset$ holds for some $T-\mathcal{S}$-fragment F. Then neither 1. nor 3. of Lemma 5 holds, and, since A is a proper end, 4. does not hold. By 2. of Lemma 5, $|A| > |A \cap T| >^* |\overline{F} \cap T| = |\overline{F}|$, which contradicts the fact that A is an $\mathcal{S}$-atom. Hence $A \subseteq T$. If $F \cap \overline{A} \neq \emptyset$ then $|F \cap N_G(A)| \geq^* |A \cap T| = |A|$, and if, otherwise, $F \cap \overline{A} = \emptyset$ then $|F \cap N_G(A)| = |F| \geq |A|$ since A is an $\mathcal{S}$-atom. So $|F \cap N_G(A)| \geq |A|$ in either case, and, by symmetry, $|\overline{F} \cap N_G(A)| \geq |A|$. Hence $|N_G(A) - T| = |F \cap N_G(A)| + |\overline{F} \cap N_G(A)| \geq 2|A|$, which implies the assertion. ∎

A noncomplete graph G is called *ℓ-critically k-connected* if $G-X$ has connectivity $k-|X|$ for all $X \subseteq V(G)$ with $|X| \leq \ell$. Hence a graph is 1-critically k-connected if and only if it has connectivity k and $\bigcup \mathcal{T}(G) = V(G)$. In particular, every 1-critically k-connected graph is almost critically k-connected. Furthermore, every 2-critically k-connected graph is also contraction critically k-connected.

For natural numbers $a \le b$ and ℓ take disjoint sets $A_1,\dots,A_{\ell+1},B_1,\dots,B_{\ell+1}$ with $|A_j|=a$ and $|B_j|=b$ for $j\in\{1,\dots,\ell+1\}$. Let $G(a,b,\ell)$ be defined by $V(G(a,b,\ell)) := \bigcup_{j=1}^{\ell+1}(A_j \cup B_j)$ and $E(G(a,b,\ell)) := \{xy : x \in A_j \wedge y \in V(G(a,b,\ell)) - (B_j \cup \{x\})$ or $x \in B_j \wedge y \in V(G(a,b,\ell)) - (A_j \cup \{x\})$ for some $j\in\{1,\dots,\ell+1\}\}$. So $G(a,b,\ell)$ is the complementary graph of the union of $\ell+1$ disjoint copies of the complete bipartite graph $K_{a,b}$. $G(a,b,\ell)$ is an ℓ-critically $\ell\cdot(a+b)$-connected graph whose fragments are $A_1,\dots,A_{\ell+1},B_1,\dots,B_{\ell+1}$.

The graph $G(1,1,\ell)$ is an ℓ-critically 2ℓ-connected graph, and removing a single vertex from an ℓ-critically k-connected graph produces an $(\ell-1)$-critically $(k-1)$-connected graph. Thus, for $\ell \le k/2$, there always exists an ℓ-critically k-connected graph.

Now we turn our attention to the following generalization of Theorem 6 in [16] and of Theorem 3 in [19]. We follow the lines of the proof in [15], with a short cut relying on Theorem 3.

Theorem 5. *Let $\mathcal{S}$ be a closed set of smallest separators in a graph G. Suppose that $W \subseteq \bigcup\mathcal{S}$ intersects every $\mathcal{S}$-fragment.*

Then there exist four $\mathcal{S}$-fragments F_1,F_2,F_3,F_4 such that $F_1,F_2,F_3,F_4\cap W$ are disjoint.

Proof. Suppose, to the contrary, that there are no such four fragments. Let A be an $\mathcal{S}$-atom and $T_A := N_G(A)$. Let $\mathcal{R} := \{T\in\mathcal{S} : T\cap A \ne \emptyset\}$.

Claim 1. Let C be an $\mathcal{R}$-end and suppose that $T\in\mathcal{S}$ intersects C. Then $T\notin\mathcal{R}$.

Suppose, to the contrary, that $T\in\mathcal{R}$, and set $T_C := N_G(C)\in\mathcal{R}$. By Theorem 4, $A\subseteq T$. By Lemma 5, either $C\subseteq T$ or $\overline{C}\subseteq T$ or $F\subseteq T_C$ for some T-fragment F, and we find four disjoint fragments $A,C,F,\overline{F}$ or $A,\overline{C},F,\overline{F}$ or $A,C,\overline{C},F$, respectively. This proves Claim 1.

Claim 2. $\overline{A}$ contains exactly two distinct $\mathcal{S}$-ends, and they are disjoint.

Suppose that $B,B'\subseteq\overline{A}$ are distinct $\mathcal{S}$-ends. Since $A\subseteq\overline{B}\cap\overline{B'}$, $B\cap B' =^* \emptyset$. Hence in order to prove Claim 2, it is sufficient to prove that there are two distinct $\mathcal{S}$-ends in $\overline{A}$.

Take any $\mathcal{S}$-end B in $\overline{A}$ and set $T_B := N_G(B)$. Then $T_B\notin\mathcal{R}$. There exist at least two disjoint $\mathcal{R}$-ends C_1,C_2, and we may assume $C_1\cap B\cap W\ne\emptyset$ or $C_2\cap B\cap W\ne\emptyset$, for otherwise $A,C_1,C_2,B\cap W$ would be disjoint. Without loss of generality, $C_1\cap B\cap W\ne\emptyset$, and we may take $T\in\mathcal{S}$ which intersects $C_1\cap B$ in W. By Claim 1, $T\cap A=\emptyset$, hence there exists a T-fragment F which contains A. Now $\overline{F}\subseteq\overline{A}$ contains an $\mathcal{S}$-end $B'\subseteq\overline{A}$. Since T intersects B, $B\ne B'$.

This proves Claim 2.

According to Claim 2, let B,B' be the two $\mathcal{S}$-ends contained in $\overline{A}$.

Claim 3. Let C be an $\mathcal{S}$-end distinct from A, B, B' such that $B \cap C \cap W \neq \emptyset$. Then $B' \cap \overline{C} = \emptyset$.

Let $T_C := N_G(C)$, $T_B := N_G(B)$, and $T_{B'} := N_G(B')$. Since $\overline{B} \cap \overline{C} =^* \emptyset$ we obtain $|B| > |\overline{C}|$ by Lemma 4.

Since $A \neq C \nsubseteq \overline{A}$, $T_C \cap A \neq \emptyset$ follows, hence $A \subseteq T_C$, and C is an $\mathcal{R}$-end as well. There exists a $T \in \mathcal{S}$ which intersects $B \cap C \cap W$. By Claim 1, $T \cap A = \emptyset$. Hence there exists a T-fragment F containing A, hence $\overline{F} \subseteq \overline{A}$ contains one of B, B'. Since T intersects B, $B' \subseteq \overline{F}$ follows. Now $A \subseteq F \cap \overline{B}$, hence $\overline{F} \cap B =^* \emptyset$, which implies $|F| > |B|$ by Lemma 4.

Suppose, to the contrary, that $B' \cap \overline{C} \neq \emptyset$. Then $\overline{F} \cap \overline{C} \neq \emptyset$, and since T intersects C, we obtain $F \cap C =^* \emptyset$, and, by Lemma 4, $|\overline{C}| > |F|$. This contradicts $|F| > |B| > |\overline{C}|$, hence Claim 3 is proved.

Now suppose that there exists an $\mathcal{S}$-end C of G distinct from A, B, B'. Then either $B \cap C \cap W \neq \emptyset$ or $B' \cap C \cap W \neq \emptyset$, for otherwise $A, B, B', C \cap W$ would be disjoint. Without loss of generality, $B \cap C \cap W \neq \emptyset$. By Claim 3, $B' \cap \overline{C} = \emptyset$. We obtain $B' \cap C \cap W \neq \emptyset$, for otherwise $A, B', \overline{C}, C \cap W$ would be disjoint. Since we may swap the roles of B and B' in Claim 3, $\overline{C} \cap B = \emptyset$ follows, and, thus, $A, B, B', \overline{C}$ are disjoint.

Hence A, B, B' are the unique $\mathcal{S}$-ends of G. We take $a \in A \cap W$, $b \in B \cap W$, $b' \in B' \cap W$. Clearly, $W' := \{a, b, b'\}$ is an $\mathcal{S}$-end cover, and, for every $u \in W'$, there exists a $T \in \mathcal{S}$ containing u. Hence the conditions of Theorem 3 are fullfilled for W' instead of W and for $\ell = 1$, implying that $|W'| \geq 4$ – a contradiction. ∎

By applying Theorem 5 and Lemma 3 one easily derives the following four corollaries.

Corollary 1. *[16, Theorem 6] Every almost critically k-connected graph G admits four fragments F_1, F_2, F_3, F_4 such that $F_1, F_2, F_3, F_4 \cap \bigcup \mathcal{T}(G)$ are disjoint.*

Corollary 2. *[19, Theorem 3] For every end B of a contraction critically k-connected graph G there exist four fragments F_1, F_2, F_3, F_4 whose neighborhoods intersect B such that $F_1, F_2, F_3, F_4 \cap (N_G(B) \cup B)$ are disjoint.*

Corollary 3. *For every end B of a 2-critically k-connected graph there exist four disjoint fragments whose neighborhoods intersect B.*

From Lemma 6 and Corollary 1 it follows that every almost critically k-connected graph without fragments of cardinality less than $k/2$ admits four disjoint fragments of cardinality $k/2$, which generalizes results from [12], [15], and [17]. For a discussion and a more general statement (concerning the local relationship of the four fragments), see [8].

Lots of further results on fragments in graphs generalize to $\mathcal{S}$-fragments. Just to mention another one, if $\mathcal{S}$ is a closed separator set of some graph G then the number of its proper $\mathcal{S}$-ends is at least the number of its non-proper $\mathcal{S}$-ends (cf. [10]).

5. Small fragments in contraction critically k-connected graphs

This section is devoted to applications of the previous results to contraction critically k-connected graphs. A key observation in this area is that removing an atom A of such a graph results in an almost critically $(k-|A|)$-connected graph. This does not generalize to ends, we can neither control connectivity nor distribution of smallest separators in the reductions. The previously developped theory will contribute here by providing results on the distribution of disjoint fragments "close to a prescribed end".

In generalizing a result of Thomassen [21], Mader proved that the number of triangles in a contraction critically k-connected graph G is at least $\frac{1}{3}|V(G)|$ [16]. In Theorem 6 we prove the analogous result for the number of disjoint fragments of cardinality at most $k/4$, i. e. for every k there exists a constant $c_k > 0$ such that every k-contraction critical graph G has at least $c_k \cdot |V(G)|$ disjoint fragments of cardinality at most $k/4$.

It is not too hard to give an *ad hoc* existence proof, but a good constant remains of course a matter of interest. The following lemma is the key step towards this result.

Lemma 7. *Let B be a proper end in a contraction critically k-connected graph. Then either $N_G(B)$ contains an end of cardinality at most $k/4$ or $|B| \le k/4$.*

Proof. Set $\mathcal{S} := \{T \in \mathcal{T}(G) : B \cap T \neq \emptyset\}$. Since G is contraction critical, $W := B \cup N_G(B) \subseteq \bigcup \mathcal{S}$. From Lemma 1 it follows easily that every $\mathcal{S}$-fragment intersects W. By Theorem 5, there exist four $\mathcal{S}$-fragments F_1, F_2, F_3, F_4 which do not intersect in W. Hence there exists a $j \in \{1,2,3,4\}$ such that $|F_j \cap N_G(B)| \le k/4$. If $F_j \subseteq N_G(B)$ or 2. of Lemma 5 (with F_j for F) holds for F_j then the assertion is proved. Otherwise, 3. of Lemma 5 holds for $T := N_G(F_j)$, and $F_j \cap \overline{B} \neq \emptyset$, implying that $k/4 \ge |F_j \cap N_G(B)| \ge^* |B \cap T| = |B|$. ∎

In particular, every contraction critically k-connected graph has a fragment of cardinality at most $k/4$, which first has been proved in [5]. As the graphs $G(\lfloor k/4 \rfloor, k-3\lfloor k/4 \rfloor, 2)$ (defined in the paragraphs following Theorem 4) show, the cardinality bound cannot be improved here.

Lemma 8. *For every vertex x of a k-contraction critical graph G there exists a proper end B_x of cardinality at most $k/4$ and an x,B_x-path of length at most $\frac{k-1}{2}$ whose inner vertices have degree at most $\frac{3}{2}k-1$.*

Proof. Set $\mathcal{S}:=\{T\in\mathcal{T}(G): x\in T\}$ and consider an $\mathcal{S}$-atom A. By definition, A is proper, and by Theorem 4, $|A|\leq\frac{k-1}{2}$. So every vertex in A has degree at most $k+|A|-1\leq\frac{3}{2}k-1$. Consider an end B contained in A. Then B is proper and $|B|\leq\frac{k-1}{2}$, too.

If B has cardinality at most $k/4$ then it serves for B_x, since there exists a path of length at most $|A|$ from x to every vertex in A whose inner vertices are in A.

Otherwise, by Lemma 7, $N_G(B)$ contains a fragment B_x of cardinality at most $k/4$. If $x\in B_x$ then B_x satisfies the statement of the assertion for trivial reasons, and if B_x intersects A then there exists an x,B_x-path of length at most $|A|$ whose inner vertices are in A. Otherwise, $B_x\subseteq T_A$ has at least two neighbors in A, since $|A|\geq|B|>k/4\geq|B_x|\geq 1$ and A is an $\mathcal{S}$-atom. Hence there is a path of length at most $|A|-2$ in A connecting a neighbor of x with a neighbor of B_x, which implies that there is an x,B_x-path of length $|A|$ whose inner vertices are in A. ∎

Theorem 6. *Every contraction critically k-connected graph G has at least $c_k\cdot|V(G)|$ disjoint ends of cardinality at most $k/4$, where*

$$c_k := \left(\left\lfloor\frac{5k}{4}\right\rfloor + k\left\lfloor\frac{3}{2}k-2\right\rfloor^{\lfloor\frac{k-3}{2}\rfloor}\right)^{-1}$$

Proof. Consider a fixed proper end B of cardinality at most $k/4$. B has exactly k neighbors, and these have at least on neighbor in B. Then we can estimate the maximal number of vertices x such that there exists a B,x-path of length at most $\ell:=\lfloor\frac{k-1}{2}\rfloor$ whose inner vertices have degree at most $d:=\lfloor\frac{3}{2}k-1\rfloor$ by

$$\begin{aligned}
d_k &:= \left\lfloor\frac{k}{4}\right\rfloor + k + k(d-1) + k(d-2)(d-1) + \ldots + k(d-2)(d-1)^{\ell-2}\\
&= \left\lfloor\frac{k}{4}\right\rfloor + 2k + k(d-2)(1+(d-1)^2+\ldots+(d-1)^{\ell-2})\\
&= \left\lfloor\frac{k}{4}\right\rfloor + 2k + k(d-2)((d-1)^{\ell-1}-1)/(d-2)\\
&= \left\lfloor\frac{k}{4}\right\rfloor + k(d-1)^{\ell-1}\\
&= \left\lfloor\frac{5k}{4}\right\rfloor + k\left\lfloor\frac{3}{2}k-2\right\rfloor^{\lfloor\frac{k-3}{2}\rfloor}.
\end{aligned}$$

Provided that G has b many proper ends of cardinality at most $k/4$, we thus obtain $|V(G)| \leq bd_k$, which implies the assertion. ∎

In particular, $c_4 = 1/6$, $c_5 = 1/27$, $c_6 = 1/44$, $c_7 = 1/450$. For $k \leq 6$, the conclusion in Theorem 6 remains true for much much smaller constants c_k ($c_4 = 1$ (every contraction critically 4-connected graph is 4-regular (cf. [14]), $c_5 = 1/5$ [2,9], $c_6 = 1/6$ [1]). However, the existence of c_7 seemed not to be known (cf. [1]).

It is known that every contraction critically 7-connected graph has two vertices of degree 7 (and an open problem whether it has even two adjacent vertices of degree 7) [8]. We will now generalize this result.

We start by proving an analogue of Lemma 7 for *non-proper* ends.

Lemma 9. *Let B be a nonproper end in a contraction critically k-connected graph G. Then either $N_G(B)$ contains an end of cardinality at most $k/4$, or $|\overline{B}| < k/4$ and there exists an end $C \neq B$ disjoint from $\overline{B}$ such that $|B| + |C| \leq \frac{k}{2} - 2$.*

Proof. Set $\mathcal{S} := \{T \in \mathcal{T}(G) : T \cap B \neq \emptyset\}$ and $T_B := N_G(B)$. Since G is contraction critically k-connected, $B \cup T_B \subseteq \bigcup \mathcal{S}$. Since every $\mathcal{S}$-fragment intersects T_B, we may apply Theorem 4 (with $W = T_B$). Hence there exists an $\mathcal{S}$-end A such that $|A \cap T_B| \leq \lfloor k/4 \rfloor$. If $A \subseteq T_B$ or 2. of Lemma 5 holds (applied to A for F) then A or $\overline{A}$ is a fragment contained in $N_G(B)$ of cardinality at most $k/4$.

Hence we may assume that 4. of Lemma 5 holds and $A \cap B \neq \emptyset$. Setting $T_A := N_G(A)$, we obtain $\lfloor k/4 \rfloor \geq |A \cap T_B| \geq |\overline{B} \cap T_A| + 1 = |\overline{B}| + 1$. In particular, $|A| \geq 3$, and $k \geq 8$.

Claim 1. $\overline{B} \subseteq T$ for every $T \in \mathcal{S}$.

For otherwise, $F \cap \overline{B} \neq \emptyset$ for some T-fragment F. By Lemma 5, $\overline{F} \subseteq T_B$. We may assume $|\overline{F}| > k/2 - 3 - |\overline{B}|$, for otherwise $\overline{F}$ serves for C. Hence $|\overline{B}| \geq |\overline{B} \cap T| + 1 \geq^* |\overline{F} \cap T_B| + 1 = |\overline{F}| + 1 > k/2 - 2 - |\overline{B}|$, so $|\overline{B}| > k/4 - 1$, a contradiction. This proves Claim 1.

By Claim 1, the $\mathcal{S}$-fragments of G are the fragments of $G - \overline{B}$, and $\mathcal{T}(G - \overline{B}) = \{T - \overline{B} : T \in \mathcal{S}\}$. We may assume that $|C| > k/2 - 2 - |\overline{B}| =: a$ holds for every $\mathcal{S}$-fragment C, otherwise the statement is proved.

Now set $\mathcal{R} := \{T \in \mathcal{S} : T \cap A \neq \emptyset\}$.

Claim 2. $|F \cap T_A| > a$ for every $T - \mathcal{R}$-fragment F.

This is true if $F \subseteq T_A$ or 2. of Lemma 5 (applied to $G - \overline{B}$, A for G, B) holds. Hence we may assume that $F \not\subseteq T_A$ and either 3. or 4. of Lemma 5 (applied just as before) holds. If 3. holds then $A \subseteq T$, so $F \cap \overline{A} \neq \emptyset$ and

$|F\cap T_A|\geq^* |A\cap T|=|A|>a$, otherwise $\overline{A}\subseteq T$ and $F\cap A\neq\emptyset$, so $|F\cap T_A|>^* |\overline{A}\cap T|=|\overline{A}|>a$.

Set $M:=X_G(A,B)-N_G(A\cap B)$.

Claim 3. $|M|\leq k/4-|\overline{B}|-1$.

Since $X(A,B)=M\cup N_G(A\cap B)$ we obtain $|X(A,B)|\geq|M|+k+1$. Since $|X(A,B)|+|T_B-A|+|\overline{B}|=2k$, we obtain $\frac{k}{4}\geq|A\cap T_B| = k-|T_B-A| = k-2k+|\overline{B}|+|X(A,B)| \geq k-2k+|\overline{B}|+|M|+k+1$, so $|M|\leq k/4-|\overline{B}|-1$, which proves Claim 3.

Set $\mathcal{Q}:=\{T\in\mathcal{S}: T\cap A\cap B\neq\emptyset\}$. Clearly, $\mathcal{Q}$ is a nonempty subset of $\mathcal{R}$.

Claim 4. $\mathcal{Q}$ is closed.

Suppose, to the contrary, that there are intersecting $\mathcal{Q}$-fragments F,F' such that equality holds in (1). Then $L:=X(F,F')\in\mathcal{T}(G)-\mathcal{Q}$ and $L=N_G(F\cap F')=N_G(\overline{F\cap F'}) = N_G(\overline{F}\cup\overline{F'})$. If $A\cap B\subseteq F\cup T$ or $A\cap B\subseteq F'\cup T'$ holds then the constadiction is immediate. Hence both $\overline{F}$ and $\overline{F'}$ intersect $A\cap B$, which implies $(F\cup F')\cap(\overline{A}\cup\overline{B})=\emptyset$. The nonempty set $\overline{F\cap F'}\cap A\cap B$ has at least $k+1$ neighbors in G, since it's a proper subset of the end B, and no neighbor in $A\cap B$, for such a neighbor would be in L.

Hence $|(\overline{F\cap F'}\cup L)\cap X(A,B)|\geq k+1$. If $F\cap F'\subseteq X(A,B)$ then $|X(A,B)|> k+1+a$ trivially. Otherwise, $F\cap F'\cap A\cap B\neq\emptyset$ follows, so $\overline{A}\cup\overline{B}\subseteq^* T\cap T'\subseteq L$, implying that $|F\cap F'\cap X(A,B)|\geq|(F\cap F')\cap(T_A-\overline{B})| = |(F\cap F')\cap T_A|\geq^* |\overline{A}\cap L|=|\overline{A}|>a$, so $|X(A,B)|>k+1+a$, too. Since $|X(A,B)|+|T_B-A|+|\overline{B}|=2k$, we derive $|T_B-A|+|\overline{B}|<k-1-a=k-1-k/2+2+|\overline{B}|$, so $|T_B-A|<k/2+1$. However, $|T_B\cap A|\leq k/4$, so $k=|T_B|<\frac{3}{4}k+1$, which is absurd. This proves Claim 4.

Claim 5. Every $\mathcal{Q}$-fragment F intersects $N_G(A\cap B)$.

Let F be a T-fragment where $T\in\mathcal{Q}$. Then there exist an $s\in T\cap A\cap B$, a $t\in T\cap\overline{B}\subseteq T\cap\overline{A\cap B}$, and an s,t-path P of length at least 2 whose inner vertices are in F. Since P intersects $N_G(A\cap B)$, so does F. This proves Claim 5.

Since G is contraction critically k-connected, $W:=N_G(A\cap B)\subseteq\bigcup\mathcal{Q}$ follows. Now we are in a situation where we may apply once again Theorem 4. By Claims 4 and 5 we may apply it to $\mathcal{Q}$, and obtain four $\mathcal{Q}$-fragments F_1,F_2,F_3,F_4 such that $F_1,F_2,F_3,F_4\cap W$ are disjoint.

Since $F_4\cap W\cap T_A = F_4\cap T_A-M$ we obtain by Claim 1 to Claim 4 that $k=|T_A|\geq|\overline{B}|+|F_1\cap T_A|+|F_2\cap T_A| + |F_3\cap T_A|+|F_4\cap T_A-M|\geq |\overline{B}|+4\cdot(\lfloor k/2\rfloor-1-|\overline{B}|) - \lfloor k/4\rfloor+|\overline{B}|+1 = 4\lfloor k/2\rfloor-3\lfloor k/4\rfloor-3$.

This inequality can only be true if $k=9$, $M=\emptyset$, $|\overline{B}|=1$ and $|F_j\cap T_A|=2$ for $j\in\{1,2,3,4\}$. We may assume that $\overline{B}$ is the unique fragment of cardinality 1 in G. Let's assume for a while that $F_j\not\subseteq T_A$ for some F_j. Then 2. of

Lemma 5 (with F_j, A for F, B) does not hold, since $|\overline{B}|$ is the only fragment of cardinality 1 in G. Since 1. does not hold either, 3. or 4. holds. If $A \subseteq N_G(F_j)$ then $F_j \cap \overline{A} \neq \emptyset$, so $|F_j \cap T_A| \geq^* |A \cap N_G(F_j)| = |A| \geq 3$, and, otherwise, if $\overline{A} \subseteq N_G(F_j)$ then $F_j \cap A \neq \emptyset$ and $|F_j \cap T_A| >^* |\overline{A} \cap N_G(F_j)| = |\overline{A}| \geq 2$ which is not possible. Hence all F_j are contained in T_A and, thus, disjoint fragments of cardinality 2 in G. Since $\overline{B}$ is the only fragment of cardinality 1 in G, they are contained in T_B as well, which implies $|T_B| \geq 8 + |A \cap T_B| \geq 10$, a contradiction. ∎

Theorem 7. *Every contraction critically k-connected graph G admits two disjoint fragments A, B such that $|A|+|B| \leq 2\lfloor k/4 \rfloor$.*

Proof. By Lemma 7, there exists a fragment A of cardinality at most $k/4$. Consider an end B in $\overline{A}$. If $|B| \leq k/4$ then the assertion is proved. If $N_G(B)$ contains an end of cardinality at most $k/4$ then it is proved, too. If B has none of these properties then it must be nonproper by Lemma 7 and, by Lemma 9, there exists an end C disjoint from $\overline{B}$ such that $|B|+|C| \leq k/2-2$, proving our Theorem. ∎

In [19], Su proved that every contraction critically k-connected graph admits two disjoint fragments A, B such that $|A|+|B| \leq k/2$. For $k \equiv 2 \bmod 4$ or $k \equiv 3 \bmod 4$, Theorem 7 improves this by one. Again, the graphs $G(\lfloor k/4 \rfloor, k-3\lfloor k/4 \rfloor, 2)$ show that the bound in Theorem 7 cannot be improved. For a further discussion, I would like refer to the very last section.

6. On the structure of contraction critically k-connected graphs of maximal minimum degree

This section is devoted to a conjecture of Su [19] on contraction critically k-connected graphs having minimum degree $5k/4-1$ (by Theorem 7, this is the largest minimum degree a k-contraction critical graph can have). He conjectured that such a graph has at least $3/2k$ vertices of degree $5k/4-1$. Here we will prove that such a graph has six disjoint fragments of cardinality $k/4$, from which Su's conjecture follows easily.

We start with a somehow surprising observation on the ends in contraction critical graphs whitout fragments of cardinality less than $k/4$.

Lemma 10. *Let G be a contraction critically k-connected graph without fragments of cardinality less than $k/4$. Then every end of G has cardinality $k/4$.*

Proof. Let B be a T_B-end of G, and suppose, to the contrary, that $|B|>k/4$.

Let $\mathcal{R}:=\{T\in\mathcal{T}(G): T\cap B\neq\emptyset\}$. By Theorem 4, there exist $\mathcal{R}$-fragments A_1,A_2,A_3,A_4 such that $A_1,A_2,A_3,A_4\cap\bigcup\mathcal{R}$ are disjoint. We apply Lemma 5 to A_j for F. If 1. holds then $|A_j\cap T_B|\geq k/4$, if 2. holds then $|A_j\cap T_B|>k/4$. If 3. holds but 1. does not then $|A_j\cap T_B|\geq|B|>k/4$, and if 4. holds but 1. does not then $|A_j\cap T_B|>|\overline{B}|\geq k/4$. Hence $|A_j\cap T_B|\geq k/4$ in either case. On the other hand, $\sum_{j=1}^{4}|F_j\cap T_B|\leq|T_B|=k$, implying that $|A_j\cap T_B|=k/4$ and $A_j\subseteq T_B$ for all $j\in\{1,2,3,4\}$. Hence T_B is the union of the four disjoint atoms A_1,A_2,A_3,A_4.

Suppose that there exists a $T\in\mathcal{R}$ intersecting two of the A_j, say A_2 and A_3. Since the A_j are atoms of G, there exists a T-fragment F such that $F\cap T_B=A_1$, $T\cap T_B=A_2\cup A_3$, and $\overline{F}\cap T_B=A_4$. Neither 1. nor 2. in Lemma 5 holds. If 3. holds then $|T|\geq 2|B|+|A_2\cup A_3|$ follows, contradicting $|B|>k/4$, and if 4. holds then $|T|>2|\overline{B}|+|A_2\cup A_3|\geq k$ follows, again a contradiction.

Hence every $T\in\mathcal{R}$ intersects at most one of the A_j. There exists a $T\in\mathcal{R}$ intersecting A_2. It follows that there exists a T-fragment F such that F intersects precisely one of A_1,A_3,A_4. Without loss of generality, F intersects A_1, and, thus, $F\cap T_B=A_1$, $T\cap T_B=A_2$, and $\overline{F}\cap T_B=A_3\cup A_4$. If $F\cap B\neq\emptyset$ then $\overline{F}\cap\overline{B}=^*\emptyset$ and $k/4=|F\cap T_B|>^*|\overline{B}\cap T|$. Hence $|X(F,\overline{B})|<\frac{3}{4}k$, so $\overline{B}$ is contained in T and has less than $k/4$ vertices, a contradiction. It follows $F\cap B=\emptyset$. Furthermore, $F\cap\overline{B}=\emptyset$ (so $F=A_1$), for otherwise $B\subseteq^* T$, which would $|\overline{B}\cap T|\geq^*|\overline{F}\cap T_B|=k/2$, so $|B|\leq k/4$.

Let $\mathcal{S}:=\{T'\in\mathcal{T}(G): T'\cap F\neq\emptyset\}$. By Theorem 4, there exist $\mathcal{S}$-fragments F_1,F_2,F_3,F_4 such that $F_1,F_2,F_3,F_4\cap\bigcup\mathcal{S}$ are disjoint. Note that A_2 is an $\mathcal{S}$-atom, so we may assume $F_1=A_2$ without loss of generality. Similarly as above, $|F_j\cap T|=k/4$. Since $|B\cap T|>^* k/4$, $B\cap T$ intersects at least two of F_2,F_3,F_4. Without loss of generality, let these be F_3 and F_4. Now $F_3\cap B$ is a proper subset of the end B, which implies $|F_3\cap T_B|>^*|\overline{B}\cap N_G(F_3)|$ and $|B\cap N_G(F_3)|>^*|\overline{F_3}\cap T_B|\geq 0$, so $N_G(F_3)\in\mathcal{R}$. In particular, F_3 contains exactly one of A_2,A_3,A_4, and so does F_4. Since $T_B\in\mathcal{S}$, the $F_j\cap T_B$ are disjoint, and we may assume $F_3\cap T_B=A_3$ and $F_4\cap T_B=A_4$ without loss of generality.

We have $F_3\cap\overline{B}=\emptyset$, for otherwise $\overline{F_3}\subseteq^* T_B$ and $|F_3\cap T_B|>^*|\overline{B}\cap N_G(F_3)|\geq|\overline{F_3}\cap T_B|=|\overline{F_3}|\geq k/4$, which is absurd. Hence $\overline{B}\subseteq N_G(F_3)$, which yields again $|F_3\cap T_B|>^*|\overline{B}\cap N_G(F_3)|\geq k/4$ – a contradiction. ∎

Theorem 8. *Let G be a contraction critically k-connected graph without fragments of cardinality less than $k/4$.*

Then every atom is adjacent to at least two other atoms and G has at least six disjoint atoms.

Proof. Let A be a T_A-atom of G and let $\mathcal{S} := \{T \in \mathcal{T}(G) : A \cap T \neq \emptyset\}$. By Theorem 5, there exist $\mathcal{S}$-ends F_1, F_2, F_3, F_4 such that $F_1, F_2, F_3, F_4 \cap \bigcup \mathcal{S}$ are disjoint. Let $T_j := N_G(F_j)$ for $j \in \{1,2,3,4\}$. As we have seen before, $|F_j \cap T_A| = k/4$ must hold. In particular, if $|F_j| < k/2$ then $F_j \subseteq T_A$ and $|F_j| = k/4$.

Claim 1. If F is a proper $T - \mathcal{S}$-fragment of G such that $k/4 < |F| < k/2$ then T_A contains a pair of adjacent atoms $B \subseteq F$, $C \subseteq T$.

By Lemma 10, F contains exactly one atom B, which, in addition, intersects T_A. Without loss of generality, $B = F_1$. If $F \cap \overline{A} \neq \emptyset$ then $|F \cap T_A| >^* |A| = k/4$, and if $F \subseteq T_A$ then $|F \cap T_A| > k/4$ trivially. Hence F intersects one of F_2, F_3, F_4 (in T_A) as well. Without loss of generality, $F \cap F_2 \neq \emptyset$, and it follows $\overline{F} \cap \overline{F_2} =^* \emptyset$. From $F \cap \overline{F_2} \neq \emptyset$ it would follow $|F| > |F \cap T_2| \geq^* |F_2 \cap T| >^* |\overline{F} \cap T_2| = |\overline{F}|$, a contradiction. Hence $\overline{F_2} \subseteq T$.

We have $|\overline{F_2}| < k/2$ as well, for otherwise $|F \cap T_2| >^* |\overline{F_2} \cap T| = |\overline{F_2}| \geq k/2$, which implies $\overline{F} \subseteq^* T_2$ and $|\overline{F} \cap T_2| < k/4$, a contradiction.

Hence $\overline{F_2}$ contains precisely one atom C of G, which, in addition, intersects T_A. It follows $C \subseteq T_A$. Since C has at least $|C| = k/4$ neighbors in F and since $|F| < k/2$, C has a neighbor in B, proving Claim 1.

Claim 2. If $\overline{F_j}$ is an atom then it is adjacent to some atom in T_A.

Without loss of generality, let $F_j = F_2 = \overline{F_1}$. So $(F_3 \cup F_4) \cap T_A \subseteq T_2$ (hence $|T_A \cap T_2| \geq k/2$), and $F := F_2 \cap \overline{A}$ is either empty or a T-fragment, where $T = X(F_2, \overline{A})$. Since $T_2 \in \mathcal{S}$ and $T_A \subseteq \bigcup \mathcal{S}$, the F_j do not intersect in $T_2 \cup T_A$. Furthermore, T_3 and T_4 both intersect the atoms A and $\overline{F_2}$, so $A \cup \overline{F_2} \subseteq T_3 \cap T_4$. If $F_3 \cap F \neq \emptyset$ then $|F_3 \cap T| \geq^* |\overline{F} \cap T_3| = k/2$, so $|F_4 \cap T| \leq k/4$, which implies that $F_4 \subseteq T_2$ is an atom adjacent to $\overline{F_2}$. Hence we may assume $F_3 \subseteq T$ and, by symmetry, $F_4 \subseteq T$. Since $|T - F_2| = \frac{3}{4}k < k$, one of F_3 or F_4 has less than $k/2$ vertices and thus must be an atom in T_A adjacent to $\overline{F_2}$. This proves Claim 2.

Claim 3. If $T \in \mathcal{S}$ intersects F_j then $|F_j| < k/2$ or $|\overline{F_j}| < k/2$, or there exists a T-fragment with less than $k/2$ vertices.

Take an arbitrary T-fragment F and suppose, to the contrary, that $F_j, \overline{F_j}, F, \overline{F}$ have at least $k/2$ vertices each. In order to apply Lemma 5 to $G - A$, we consider F as a $(T - A)$-fragment of $G - A$ and $B := F_j$ as an end of $G - A$. If 1. or 2. of 5 holds then $|N_{G-A}(B)| > k$, and if 3. or 4. holds then $|T - A| \geq k$, which is all absurd. This proves Claim 3.

Claim 4. If $T \in \mathcal{S}$ intersects F_j then either F_j is an atom, or there exists a T-atom, or there exist a pair of adjacent atoms in T_A.

If one of F_j, $\overline{F_j}$ would be a proper fragment on less than $k/2$ vertices then the assertion follows from Claim 1 or Claim 2. Hence $|F_j| \geq k/2$ and

$|\overline{F_j}| \geq k/2$. By Claim 3 there exists a proper T-fragment F on less than $k/2$ vertices. If it is not an atom then $k/4 < |F| < k/2$, and there exists a pair of adjacent atoms in T_A by Claim 1. This proves Claim 4.

Claim 5. T_A contains two (disjoint) atoms.

Suppose, to the contrary, that F_1 is the unique atom in T_A and consider any $x \in T_A - F_1$. There exists a $T \in \mathcal{S}$ containing x, so T intersects F_j for some $j \in \{2,3,4\}$. By Claim 4, there exists a T-atom, which can only be F_1. So $T_A - F_1 \subseteq T_1$, implying that $T_1 = (T_A - F_1) \cup A$. This is absurd since F_1 must have neighbors in $\overline{A}$, proving Claim 5.

Claim 6. Either T_A contains a pair of adjacent atoms or G contains six disjoint atoms.

We may assume that at least one F_j is not an atom, for otherwise F_1, F_2, F_3, F_4, A, and any end in $\overline{A}$ would form a system of six disjoint atoms. Without loss of generality, let $j = 4$. By Claim 4 we may assume that T_4 contains an atom, say F_1.

If T_4 intersects some F_j then we may assume by Claim 4 that F_j is an atom. It follows that the intersection of T_A and T_4 consists of one or two atoms and that, in either case, $F := F_4 \cap \overline{A}$ is a T-fragment, where $T = X(F_4, \overline{A})$. Without loss of generality we may assume that F_3 does not intersect T_4. Note that F_3 and F_4 are disjoint for otherwise $|F_3 \cap T_4| >^* 0$. So $F_3 \subseteq \overline{F_4}$.

Let B be a T_B-atom in F. By Claim 5, T_B contains two atoms F_5, F_6. If F_j is not contained in T_B for some $j \in \{1,2\}$ then A, B, F_j, F_5, F_6, and any atom contained in F_3 would form a system of six disjoint atoms.

Hence we may assume that every atom in F is adjacent to both F_1 and F_2. In particular, $F_2 \subseteq T_4$. By Claim 3 we may assume that $\overline{F_4}$ is not an atom, implying that $F' := \overline{F_4} \cap \overline{A}$ is a T'-fragment, where $T' := X_G(\overline{F_4}, \overline{A})$. Let B' be any atom in F'. If F_3 is an atom as well then A, B, B', F_1, F_2, F_3 would form a system of six disjoint atoms.

Hence we may assume that F_1, F_2 are the only atoms in T_A. Now for every $x \in T_A - (F_1 \cup F_2) = T_A - T_4$ there exists a $T \in \mathcal{S}$ containing x. By Claim 4 we may assume that T is one of T_1, T_2, implying that every $x \in T_A - T_4$ is adjacent to F_1 or F_2. Now F_1 has already $k/4$ neighbors in A and at least $k/4$ neighbors in each of F, F', so it has at most $k/4$ neighbors in $T_A - T_4$. The same holds for F_2, and this implies that F_1, F_2 each have exactly $k/4$ neighbors in each of F and F', and no neighbors in $\overline{A} \cap T_4$. So $Z := N_G(A \cup F_1) \cap F_4$ consists of exactly $k/2$ vertices. If $F_4 - Z$ is not empty then it is a fragment (since $|N(F_4 - Z)| = |(T_4 - (A \cup F_1)) \cup Z| = k$), which is contained in F and contains an end not adjacent to F_1 – a contradiction.

So $F_4 - Z$ is empty, F itself is an atom adjacent to F_1, F_2, and, by symmetry, F' is an atom adjacent to F_1, F_2. Now $\overline{A} \cap T_4$ is a fragment, since $|N_G(\overline{A} \cap T_4)| = |F \cup F' \cup (T_A - T_4)| = k$, implying that $A, F_1, F_2, F, F', \overline{A} \cap T_4$ forms a system of six disjoint fragments.

This proves Claim 6.

In order to prove that, in general, G must have six disjoint atoms, we may assume by Claim 6 that every atom is adjacent to two adjacent atoms. Let A_1, A_2, A_3 be three pairwise adjacent atoms. Their complements contain atoms B_1, B_2, B_3, and if these were pairwise distinct then they would form a system of six disjoints atoms together with A_1, A_2, A_3. Hence we may assume without loss of generality that $\overline{B_1}$ contains A_1 and A_2. B_1 is adjacent to two adjacent atoms C_1, C_2, and we may assume that one of them is equal to A_3, for otherwise $A_1, A_2, A_3, B_1, C_1, C_2$ would be an appropriate atom system. Without loss of generality, $C_1 = A_3$. But then C_1 is adjacent to B_1, C_2, A_2, A_1, so its neighborhood consists of four disjoint atoms, which form together with C_1 and any atom in $\overline{C_1}$ a system of six disjoint atoms. ∎

Corollary 4. *Every contraction critically k-connected graph contains either a vertex of degree less than $\frac{5}{4}k - 1$ or at least $\frac{3}{2}k$ vertices of degree $\frac{5}{4}k - 1$.*

The graphs $G(a, a, 2)$ show that we may neither expect more than six fragments in the assertion of Theorem 8 nor more than $\frac{3}{2}k$ vertices of degree $\frac{5}{4}k - 1$ in Corollary 4.

7. Conjectures and open problems

There is a certain lack of examples of contraction critically k-connected graphs having only a few “small” fragments. This leads to several open problems and further conjectures.

Problem 1. Does every contraction critically k-connected graph admit two disjoint fragments of cardinality at most $k/4$?

One could ask the same question for 2-critically k-connected graphs. By Theorem 6, *almost all* contraction critically k-connected graphs admit two disjoint fragments of cardinality at most $k/4$. It seems to be a hard question to improve the order of the bound c_k in Theorem 6 substantially. I would expect that there exists constants c_k satisfying the assertion in Theorem 6 such that c_k^{-1} grows polynomially in k (if not even quadratic).

Even better results could hold in highly critically connected graphs.

Problem 2. Does every ℓ-critically k-connected graph admit $\ell+1$ disjoint fragments of cardinality at most $\frac{k}{2\ell}$?

This is true for $\ell=1$ by the results of [15]. It would be best possible, considering the number $\ell+1$ of fragments in play, as it is shown by the graphs $G(a,b,\ell)$ where $a<b$. A weaker, cumulative variant of this problem would be the following, which I expect to be true.

Conjecture 1. Every ℓ-critically k-connected graph admits $\ell+1$ disjoint fragments whose cardinalities sum up to at most k.

This is true for $\ell=1$ as it has been mentioned above. If every ℓ-critically k-connected graph had a fragment of cardinality at most $\frac{k}{2\ell}$, as it has been conjectured in [14], then Conjecture 1 would follow inductively: The graph obtained from an ℓ-critically k-connected graph G by removing an atom A is $(\ell-1)$-critically $(k-|A|)$-connected. Thus, it contains ℓ disjoint fragments whose cardinalities sum up to $k-|A|$. Together with A they constitute $\ell+1$ disjoint fragments whose cardinalities sum up to at most k.

Let me finish with two conjectures on the ends in ℓ-critically k-connected graphs. As it has been mentioned, Mader conjectured that every ℓ-critically k-connected graph has a fragment of cardinality at most $\frac{k}{2\ell}$ [14]. This is best possible in the sense that for each k such that $\frac{k}{2\ell}=:a$ is a positive integer there exists an ℓ-critically k-connected graph G which contains a fragment of cardinality $\frac{k}{2\ell}$ and no fragments of smaller cardinality (take $G(a,a,\ell)$). Indeed, there seems to be no such graph G having ends of cardinality exceeding $\frac{k}{2\ell}$, which is reflected in the following conjecture.

Conjecture 2. Let G be an ℓ-critically k-connected graph without fragments of cardinality less than $\frac{k}{2\ell}$. Then every end of G has cardinality $\frac{k}{2\ell}$.

For $\ell=1$, this follows from Lemma 6, for $\ell=2$ it follows from Lemma 10, and for $\ell=k/2$ it follows from the fact that for every $\ell\geq 3$, the graph $G(1,1,\ell)$ is the unique ℓ-critically 2ℓ-connected graph [11].

Provided that every ℓ-critically k-connected graph had a system of $2\ell+2$ disjoint fragments, as it has been conjectured in [14], the following analogue of the second statement of Theorem 8 would follow from Conjecture 2.

Conjecture 3. Every ℓ-critically k-connected graph without fragments of cardinality less than $\frac{k}{2\ell}$ admits $2\ell+2$ disjoint fragments of cardinality $\frac{k}{2\ell}$.

*

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